CHAPTER 3

WAVE EQUATION AND ITS SOLUTIONS

3.1 INTRODUCTION

The electromagnetic fields of boundary-value problems are obtained as solutions to Maxwell's equations, which are first-order partial differential equations. However, Maxwell's equations are coupled partial differential equations, which means that each equation has more than one unknown field. These equations can be uncoupled only at the expense of raising their order. For each of the fields, following such a procedure leads to an uncoupled second-order partial differential equation that is usually referred to as the *wave equation*. Therefore electric and magnetic fields for a given boundary-value problem can be obtained either as solutions to Maxwell's or the wave equations. The choice of equations is related to individual problems by convenience and ease of use. In this chapter we will develop the vector wave equations for each of the fields, and then we will demonstrate their solutions in the rectangular, cylindrical, and spherical coordinate systems.

3.2 TIME-VARYING ELECTROMAGNETIC FIELDS

The first two of Maxwell's equations in differential form, as given by (1-1) and (1-2), are first-order, coupled differential equations; that is, both the unknown fields ($\mathscr{E}$ and $\mathscr{H}$) appear in each equation. Usually it is very desirable, for convenience in solving for $\mathscr{E}$ and $\mathscr{H}$, to uncouple these equations. This can be accomplished at the expense of increasing the order of the differential equations to second order. To do this, we repeat (1-1) and (1-2), that is,

$$\nabla \times \mathscr{E} = -\mathscr{M}_i - \mu \frac{\partial \mathscr{H}}{\partial t} \tag{3-1}$$

$$\nabla \times \mathscr{H} = \mathscr{J}_i + \sigma \mathscr{E} + \varepsilon \frac{\partial \mathscr{E}}{\partial t} \tag{3-2}$$

where it is understood in the remaining part of the book that σ represents the effective conductivity σ_e and ε represents ε'. Taking the curl of both sides of each of equations 3-1 and 3-2 and assuming a homogeneous medium, we can write that

$$\nabla \times \nabla \times \mathscr{E} = -\nabla \times \mathscr{M}_i - \mu\nabla \times \left(\frac{\partial \mathscr{H}}{\partial t}\right) = -\nabla \times \mathscr{M}_i - \mu\frac{\partial}{\partial t}(\nabla \times \mathscr{H}) \tag{3-3}$$

$$\nabla \times \nabla \times \mathscr{H} = \nabla \times \mathscr{J}_i + \sigma\nabla \times \mathscr{E} + \varepsilon\nabla \times \left(\frac{\partial \mathscr{E}}{\partial t}\right)$$

$$= \nabla \times \mathscr{J}_i + \sigma\nabla \times \mathscr{E} + \varepsilon\frac{\partial}{\partial t}(\nabla \times \mathscr{E}) \tag{3-4}$$

Substituting (3-2) into the right side of (3-3) and using the vector identity

$$\nabla \times \nabla \times \mathbf{F} = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F} \tag{3-5}$$

into the left side, we can rewrite (3-3) as

$$\nabla(\nabla \cdot \mathscr{E}) - \nabla^2\mathscr{E} = -\nabla \times \mathscr{M}_i - \mu\frac{\partial}{\partial t}\left[\mathscr{J}_i + \sigma\mathscr{E} + \varepsilon\frac{\partial \mathscr{E}}{\partial t}\right]$$

$$\nabla(\nabla \cdot \mathscr{E}) - \nabla^2\mathscr{E} = -\nabla \times \mathscr{M}_i - \mu\frac{\partial \mathscr{J}_i}{\partial t} - \mu\sigma\frac{\partial \mathscr{E}}{\partial t} - \mu\varepsilon\frac{\partial^2 \mathscr{E}}{\partial t^2} \tag{3-6}$$

Substituting Maxwell's equation 1-3, or

$$\nabla \cdot \mathscr{D} = \varepsilon\nabla \cdot \mathscr{E} = q_{ve} \Rightarrow \nabla \cdot \mathscr{E} = \frac{q_{ve}}{\varepsilon} \tag{3-7}$$

into (3-6) and rearranging its terms, we have that

$$\boxed{\nabla^2\mathscr{E} = \nabla \times \mathscr{M}_i + \mu\frac{\partial \mathscr{J}_i}{\partial t} + \frac{1}{\varepsilon}\nabla q_{ve} + \mu\sigma\frac{\partial \mathscr{E}}{\partial t} + \mu\varepsilon\frac{\partial^2 \mathscr{E}}{\partial t^2}} \tag{3-8}$$

which is recognized as an uncoupled second-order differential equation for $\mathscr{E}$.

In a similar manner, by substituting (3-1) into the right side of (3-4) and using the vector identity of (3-5) in the left side of (3-4), we can rewrite it as

$$\nabla(\nabla \cdot \mathscr{H}) - \nabla^2\mathscr{H} = \nabla \times \mathscr{J}_i + \sigma\left(-\mathscr{M}_i - \mu\frac{\partial \mathscr{H}}{\partial t}\right) + \varepsilon\frac{\partial}{\partial t}\left(-\mathscr{M}_i - \mu\frac{\partial \mathscr{H}}{\partial t}\right)$$

$$\nabla(\nabla \cdot \mathscr{H}) - \nabla^2\mathscr{H} = \nabla \times \mathscr{J}_i - \sigma\mathscr{M}_i - \mu\sigma\frac{\partial \mathscr{H}}{\partial t} - \varepsilon\frac{\partial \mathscr{M}_i}{\partial t} - \mu\varepsilon\frac{\partial^2 \mathscr{H}}{\partial t^2} \tag{3-9}$$

Substituting Maxwell's equation

$$\nabla \cdot \mathscr{B} = \mu\nabla \cdot \mathscr{H} = q_{vm} \Rightarrow \nabla \cdot \mathscr{H} = \left(\frac{q_{vm}}{\mu}\right) \tag{3-10}$$

into (3-9), we have that

$$\nabla^2 \mathscr{H} = -\nabla \times \mathscr{J}_i + \sigma \mathscr{M}_i + \frac{1}{\mu}\nabla(q_{vm}) + \varepsilon \frac{\partial \mathscr{M}_i}{\partial t} + \mu\sigma \frac{\partial \mathscr{H}}{\partial t} + \mu\varepsilon \frac{\partial^2 \mathscr{H}}{\partial t^2} \tag{3-11}$$

which is recognized as an uncoupled second-order differential equation for $\mathscr{H}$. Thus (3-8) and (3-11) form a pair of uncoupled second-order differential equations that are a by-product of Maxwell's equations as given by (1-1) through (1-4).

Equations 3-8 and 3-11 are referred to as the *vector wave equations* for $\mathscr{E}$ and $\mathscr{H}$. For solving an electromagnetic boundary-value problem, the equations that must be satisfied are Maxwell's equations as given by (1-1) through (1-4) or the wave equations as given by (3-8) and (3-11). Often, the forms of the wave equations are preferred over those of Maxwell's equations.

For source-free regions ($\mathscr{J}_i = q_{ve} = 0$ and $\mathscr{M}_i = q_{vm} = 0$), the wave equations 3-8 and 3-11 reduce, respectively, to

$$\nabla^2 \mathscr{E} = \mu\sigma \frac{\partial \mathscr{E}}{\partial t} + \mu\varepsilon \frac{\partial^2 \mathscr{E}}{\partial t^2} \tag{3-12}$$

$$\nabla^2 \mathscr{H} = \mu\sigma \frac{\partial \mathscr{H}}{\partial t} + \mu\varepsilon \frac{\partial^2 \mathscr{H}}{\partial t^2} \tag{3-13}$$

For source-free ($\mathscr{J}_i = q_{ve} = 0$ and $\mathscr{M}_i = q_{vm} = 0$) and lossless media ($\sigma = 0$), the wave equations 3-8 and 3-11 or 3-12 and 3-13 simplify to

$$\nabla^2 \mathscr{E} = \mu\varepsilon \frac{\partial^2 \mathscr{E}}{\partial t^2} \tag{3-14}$$

$$\nabla^2 \mathscr{H} = \mu\varepsilon \frac{\partial^2 \mathscr{H}}{\partial t^2} \tag{3-15}$$

Equations 3-14 and 3-15 represent the simplest forms of the vector wave equations.

3.3 TIME-HARMONIC ELECTROMAGNETIC FIELDS

For time-harmonic fields (time variations of the form $e^{j\omega t}$), the wave equations can be derived using a similar procedure as in Section 3.2 for the general time-varying fields, starting with Maxwell's equations as given in Table 1-4. However, instead of going through this process, we find, by comparing Maxwell's equations for the general time-varying fields with those for the time-harmonic fields (both are displayed in Table 1-4), that one set can be obtained from the other by replacing $\partial/\partial t \Leftrightarrow j\omega$, $\partial^2/\partial t^2 \Leftrightarrow (j\omega)^2 = -\omega^2$, and the instantaneous fields ($\mathscr{E}, \mathscr{H}, \mathscr{D}, \mathscr{B}$), respectively, with the complex fields ($\mathbf{E}, \mathbf{H}, \mathbf{D}, \mathbf{B}$) and vice versa. Doing this for the wave equations 3-8, 3-11, 3-12, and 3-13, we can write each, respectively, as

$$\nabla^2\mathbf{E} = \nabla \times \mathbf{M}_i + j\omega\mu\mathbf{J}_i + \frac{1}{\varepsilon}\nabla q_{ve} + j\omega\mu\sigma\mathbf{E} - \omega^2\mu\varepsilon\mathbf{E} \tag{3-16a}$$

$$\nabla^2\mathbf{H} = -\nabla \times \mathbf{J}_i + \sigma\mathbf{M}_i + j\omega\varepsilon\mathbf{M}_i + \frac{1}{\mu}\nabla q_{vm} + j\omega\mu\sigma\mathbf{H} - \omega^2\mu\varepsilon\mathbf{H} \tag{3-16b}$$

$$\nabla^2\mathbf{E} = j\omega\mu\sigma\mathbf{E} - \omega^2\mu\varepsilon\mathbf{E} = \gamma^2\mathbf{E} \tag{3-17a}$$

$$\nabla^2\mathbf{H} = j\omega\mu\sigma\mathbf{H} - \omega^2\mu\varepsilon\mathbf{H} = \gamma^2\mathbf{H} \tag{3-17b}$$

where

$$\gamma^2 = j\omega\mu\sigma - \omega^2\mu\varepsilon = j\omega\mu(\sigma + j\omega\varepsilon) \tag{3-17c}$$

$$\gamma = \alpha + j\beta = \text{propagation constant} \tag{3-17d}$$

$$\alpha = \text{attenuation constant (Np/m)} \tag{3-17e}$$

$$\beta = \text{phase constant (rad/m)} \tag{3-17f}$$

The constants α, β, and γ will be discussed in more detail in Section 4.3 where α and β are expressed by (4-28c) and (4-28d) in terms of ω, ε, μ, and σ.

Similarly (3-14) and (3-15) can be written, respectively, as

$$\nabla^2\mathbf{E} = -\omega^2\mu\varepsilon\mathbf{E} = -\beta^2\mathbf{E} \tag{3-18a}$$

$$\nabla^2\mathbf{H} = -\omega^2\mu\varepsilon\mathbf{H} = -\beta^2\mathbf{H} \tag{3-18b}$$

where

$$\beta^2 = \omega^2\mu\varepsilon \tag{3-18c}$$

In the literature the phase constant β is often represented by k.

3.4 SOLUTION TO THE WAVE EQUATION

The time variations of most practical problems are of the time-harmonic form. Fourier series can be used to express time variations of other forms in terms of a number of time-harmonic terms. Electromagnetic fields associated with a given boundary-value problem must satisfy Maxwell's equations or the vector wave equations. For many cases, the vector wave equations reduce to a number of scalar Helmholtz (wave) equations, and the general solutions can be constructed once solutions to each of the scalar Helmholtz equations are found.

In this section we want to demonstrate at least one method that can be used to solve the scalar Helmholtz equation in rectangular, cylindrical, and spherical coordinates. The method is known as the *separation of variables* [1, 2], and the general solution to the scalar Helmholtz equation using this method can be constructed in 11 three-dimensional orthogonal coordinate systems (including the rectangular, cylindrical, and spherical systems) [3].

The solutions for the instantaneous time-harmonic electric and magnetic field intensities can be obtained by considering the forms of the vector wave equations given either in Section 3.2 or Section 3.3. The approach chosen here will be to use those of Section 3.3 to solve for the complex field intensities **E** and **H** first. The corresponding instantaneous quantities can then be formed using the relations (1-61a) through (1-61f) between the instantaneous time-harmonic fields and their complex counterparts.

3.4.1 Rectangular Coordinate System

In a rectangular coordinate system, the vector wave equations 3-16a through 3-18c can be reduced to three scalar wave (Helmholtz) equations. First we will consider the solutions for source-free and lossless media. This will be followed by solutions for source-free but lossy media.

A. SOURCE-FREE AND LOSSLESS MEDIA

For source-free ($\mathbf{J}_i = \mathbf{M}_i = q_{ve} = q_{vm} = 0$) and lossless ($\sigma = 0$) media, the vector wave equations for the complex electric and magnetic field intensities are those given by (3-18a) through (3-18c). Since (3-18a) and (3-18b) are of the same form, let us examine the solution to one of them. The solution to the other can then be written by an interchange of **E** with **H** or **H** with **E**. We will begin by examining the solution for **E**.

In rectangular coordinates, a general solution for **E** can be written as

$$\mathbf{E}(x, y, z) = \hat{a}_x E_x(x, y, z) + \hat{a}_y E_y(x, y, z) + \hat{a}_z E_z(x, y, z) \qquad (3\text{-}19)$$

where x, y, z are the rectangular coordinates, as illustrated in Figure 3-1. Substituting (3-19) into (3-18a) we can write that

$$\nabla^2\mathbf{E} + \beta^2\mathbf{E} = \nabla^2(\hat{a}_x E_x + \hat{a}_y E_y + \hat{a}_z E_z) + \beta^2(\hat{a}_x E_x + \hat{a}_y E_y + \hat{a}_z E_z) = 0 \qquad (3\text{-}20)$$

which reduces to three scalar wave equations of

$$\nabla^2 E_x(x, y, z) + \beta^2 E_x(x, y, z) = 0 \qquad (3\text{-}20a)$$

$$\nabla^2 E_y(x, y, z) + \beta^2 E_y(x, y, z) = 0 \qquad (3\text{-}20b)$$

$$\nabla^2 E_z(x, y, z) + \beta^2 E_z(x, y, z) = 0 \qquad (3\text{-}20c)$$

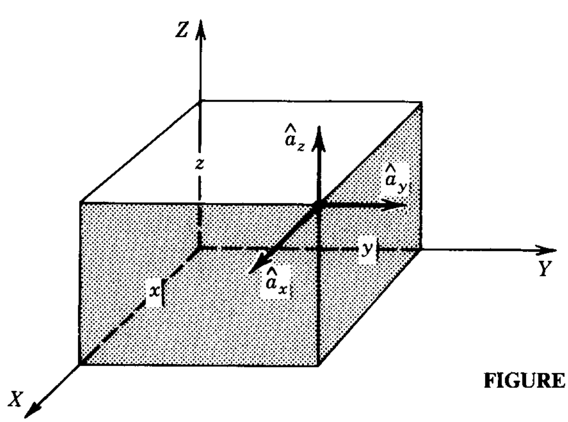

FIGURE 3-1 Rectangular coordinate system and corresponding unit vectors.

because

$$\nabla^2(\hat{a}_x E_x + \hat{a}_y E_y + \hat{a}_z E_z) = \hat{a}_x \nabla^2 E_x + \hat{a}_y \nabla^2 E_y + \hat{a}_z \nabla^2 E_z \tag{3-21}$$

Equations 3-20a through 3-20c are all of the same form; once a solution of any one of them is obtained, the solutions to the others can be written by inspection. We choose to work first with that for E_x as given by (3-20a).

In expanded form (3-20a) can be written as

$$\nabla^2 E_x + \beta^2 E_x = \frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} + \beta^2 E_x = 0 \tag{3-22}$$

Using the *separation of variables method*, we assume that a solution for $E_x(x, y, z)$ can be written in the form of

$$E_x(x, y, z) = f(x) g(y) h(z) \tag{3-23}$$

where the x, y, z variations of E_x are separable (hence the name). If any inconsistencies are encountered with assuming such a form of solution, another form must be attempted. This is the procedure usually followed in solving differential equations. Substituting (3-23) into (3-22), we can write that

$$gh \frac{\partial^2 f}{\partial x^2} + fh \frac{\partial^2 g}{\partial y^2} + fg \frac{\partial^2 h}{\partial z^2} + \beta^2 fgh = 0 \tag{3-24}$$

Since $f(x)$, $g(y)$, and $h(z)$ are each a function of only one variable, we can replace the partials in (3-24) by ordinary derivatives. Doing this and dividing each term by fgh, we can write that

$$\frac{1}{f}\frac{d^2 f}{dx^2} + \frac{1}{g}\frac{d^2 g}{dy^2} + \frac{1}{h}\frac{d^2 h}{dz^2} + \beta^2 = 0 \tag{3-25}$$

or

$$\frac{1}{f}\frac{d^2 f}{dx^2} + \frac{1}{g}\frac{d^2 g}{dy^2} + \frac{1}{h}\frac{d^2 h}{dz^2} = -\beta^2 \tag{3-25a}$$

Each of the first three terms in (3-25a) is a function of only a single independent variable; hence the sum of these terms can equal $-\beta^2$ only if each term is a constant. Thus (3-25a) separates into three equations of the form

$$\frac{1}{f}\frac{d^2 f}{dx^2} = -\beta_x^2 \Rightarrow \frac{d^2 f}{dx^2} = -\beta_x^2 f \tag{3-26a}$$

$$\frac{1}{g}\frac{d^2 g}{dy^2} = -\beta_y^2 \Rightarrow \frac{d^2 g}{dy^2} = -\beta_y^2 g \tag{3-26b}$$

$$\frac{1}{h}\frac{d^2 h}{dz^2} = -\beta_z^2 \Rightarrow \frac{d^2 h}{dz^2} = -\beta_z^2 h \tag{3-26c}$$

where, in addition,

$$\beta_x^2 + \beta_y^2 + \beta_z^2 = \beta^2 \tag{3-27}$$

Equation 3-27 is referred to as the *constraint* equation. In addition $\beta_x, \beta_y, \beta_z$ are known as the wave constants (numbers) in the x, y, z directions, respectively, that will be determined using boundary conditions.

The solution to each of equations 3-26a, 3-26b, or 3-26c can take different forms. Some typical valid solutions for $f(x)$ of (3-26a) would be

$$f_1(x) = A_1 e^{-j\beta_x x} + B_1 e^{+j\beta_x x} \tag{3-28a}$$

or

$$f_2(x) = C_1 \cos(\beta_x x) + D_1 \sin(\beta_x x) \tag{3-28b}$$

Similarly the solutions to (3-26b) and (3-26c) for $g(y)$ and $h(z)$ can be written, respectively, as

$$g_1(y) = A_2 e^{-j\beta_y y} + B_2 e^{+j\beta_y y} \tag{3-29a}$$

or

$$g_2(y) = C_2 \cos(\beta_y y) + D_2 \sin(\beta_y y) \tag{3-29b}$$

and

$$h_1(z) = A_3 e^{-j\beta_z z} + B_3 e^{+j\beta_z z} \tag{3-30a}$$

TABLE 3-1
Wave functions, zeroes, and infinities of plane wave functions in rectangular coordinates

Wave type	Wave functions	Zeroes of wave functions	Infinities of wave functions
Traveling waves	$e^{-j\beta x}$ for $+x$ travel	$\beta x \to -j\infty$	$\beta x \to +j\infty$
	$e^{+j\beta x}$ for $-x$ travel	$\beta x \to +j\infty$	$\beta x \to -j\infty$
Standing waves	$\cos(\beta x)$ for $\pm x$	$\beta x = \pm(n + \frac{1}{2})\pi$	$\beta x \to \pm j\infty$
	$\sin(\beta x)$ for $\pm x$	$\beta x = \pm n\pi$ $n = 0, 1, 2, \ldots$	$\beta x \to \pm j\infty$
Evanescent waves	$e^{-\alpha x}$ for $+x$	$\alpha x \to +\infty$	$\alpha x \to -\infty$
	$e^{+\alpha x}$ for $-x$	$\alpha x \to -\infty$	$\alpha x \to +\infty$
	$\cosh(\alpha x)$ for $\pm x$	$\alpha x = \pm j(n + \frac{1}{2})\pi$	$\alpha x \to \pm\infty$
	$\sinh(\alpha x)$ for $\pm x$	$\alpha x = \pm jn\pi$ $n = 0, 1, 2, \ldots$	$\alpha x \to \pm\infty$
Attenuating traveling waves	$e^{-\gamma x} = e^{-\alpha x} e^{-j\beta x}$ for $+x$ travel	$\gamma x \to +\infty$	$\gamma x \to -\infty$
	$e^{+\gamma x} = e^{+\alpha x} e^{+j\beta x}$ for $-x$ travel	$\gamma x \to -\infty$	$\gamma x \to +\infty$
Attenuating standing waves	$\cos(\gamma x) = \cos(\alpha x)\cosh(\beta x) - j\sin(\alpha x)\sinh(\beta x)$ for $\pm x$	$\gamma x = \pm j(n + \frac{1}{2})\pi$	$\gamma x \to \pm j\infty$
	$\sin(\gamma x) = \sin(\alpha x)\cosh(\beta x) + j\cos(\alpha x)\sinh(\beta x)$ for $\pm x$	$\gamma x = \pm jn\pi$ $n = 0, 1, 2, \ldots$	$\gamma x \to \pm j\infty$

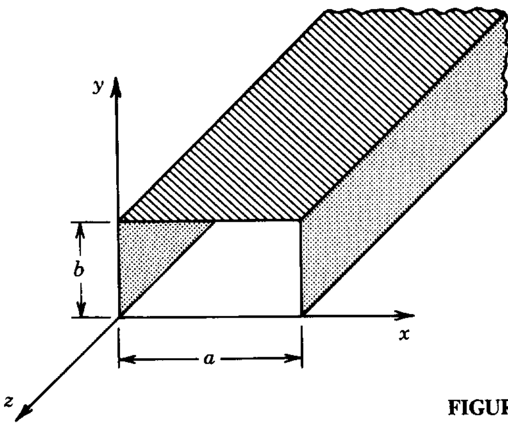

FIGURE 3-2 Rectangular waveguide geometry.

or

$$h_2(z) = C_3 \cos(\beta_z z) + D_3 \sin(\beta_z z) \tag{3-30b}$$

Although all the aforementioned solutions are valid for $f(x)$, $g(y)$, and $h(z)$, the most appropriate form should be chosen to simplify the complexity of the problem at hand. In general, the solutions of (3-28a), (3-29a), and (3-30a) in terms of complex exponentials represent *traveling waves* and the solutions of (3-28b), (3-29b), and (3-30b) represent *standing waves*. Wave functions representing various wave types in rectangular coordinates are found listed in Table 3-1. In Chapter 8 we will consider specific examples and the appropriate solution forms for $f(x)$, $g(y)$, and $h(z)$.

Once the appropriate forms for $f(x)$, $g(y)$, and $h(z)$ have been decided, the solution for the scalar function $E_x(x, y, z)$ of (3-22) can be written as the product of fgh as stated by (3-23). To demonstrate that, let us consider a specific example in which it will be assumed that the appropriate solutions for f, g, and h are given, respectively, by (3-28b), (3-29b), and (3-30a). Thus we can write that

$$E_x(x, y, z) = [C_1 \cos(\beta_x x) + D_1 \sin(\beta_x x)][C_2 \cos(\beta_y y) + D_2 \sin(\beta_y y)] \times [A_3 e^{-j\beta_z z} + B_3 e^{+j\beta_z z}] \tag{3-31}$$

This is an appropriate solution for any of the electric or magnetic field components inside a rectangular pipe (waveguide), shown in Figure 3-2, that is bounded in the x and y directions and has its length along the z axis. Because the waveguide is bounded in the x and y directions, standing waves, represented by cosine and sine functions, have been chosen as solutions for $f(x)$ and $g(y)$ functions. However, because the waveguide is not bounded in the z direction, traveling waves, represented by complex exponential functions, have been chosen as solutions for $h(z)$. A complete discussion of the fields inside a rectangular waveguide can be found in Chapter 8.

For $e^{j\omega t}$ time variations, which are assumed throughout this book, the first complex exponential term in (3-31) represents a wave that travels in the $+z$ direction; the second exponential represents a wave that travels in the $-z$ direction. To demonstrate this, let us examine the instantaneous form $\mathscr{E}_x(x, y, z; t)$ of the scalar complex function $E_x(x, y, z)$. Since the solution of (3-31) represents the

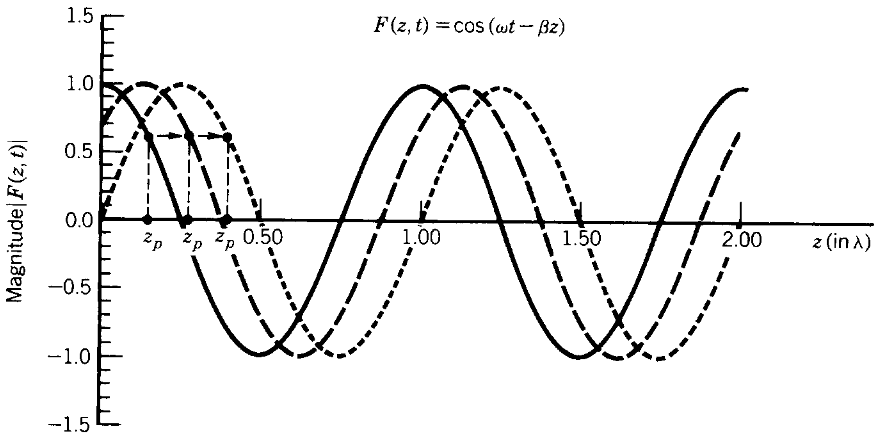

FIGURE 3-3 Variations as a function of distance for different times of positive traveling wave. —— time $t_0 = 0$; - - - - time $t_1 = T/8$; ---- time $t_2 = T/4$.

complex form of E_x, its instantaneous form can be written as

$$\mathscr{E}_x(x, y, z; t) = \text{Re}\left[E_x(x, y, z)e^{j\omega t}\right] \tag{3-32}$$

Considering only the first exponential term of (3-31) and assuming all constants are real, we can write the instantaneous form of the $\mathscr{E}_x$ function for that term as

$$\begin{aligned}\mathscr{E}_x^+(x, y, z; t) &= \text{Re}\left[E_x^+(x, y, z)e^{j\omega t}\right] \\ &= \text{Re}\left\{\left[C_1 \cos(\beta_x x) + D_1 \sin(\beta_x x)\right]\right. \\ &\qquad \left.\times\left[C_2 \cos(\beta_y y) + D_2 \sin(\beta_y y)\right] A_3 e^{j(\omega t - \beta_z z)}\right\}\end{aligned} \tag{3-33}$$

or, if the constants C_1, D_1, C_2, D_2, and A_3 are real, as

$$\begin{aligned}\mathscr{E}_x^+(x, y, z; t) &= \left[C_1 \cos(\beta_x x) + D_1 \sin(\beta_x x)\right] \\ &\qquad \times\left[C_2 \cos(\beta_y y) + D_2 \sin(\beta_y y)\right] A_3 \cos(\omega t - \beta_z z)\end{aligned} \tag{3-33a}$$

where the superscript plus is used to denote a positive traveling wave.

A plot of the normalized $\mathscr{E}_x^+(x, y, z; t)$ as a function of z for different times $(t = t_0, t_1, \ldots, t_n, t_{n+1})$ is shown in Figure 3-3. It is evident that as time increases $(t_{n+1} > t_n)$, the waveform of $\mathscr{E}_x^+$ is essentially the same, with the exception of an apparent shift in the $+z$ direction indicating a wave traveling in the $+z$ direction. This shift in the $+z$ direction can also be demonstrated by examining what happens to a given point z_p in the waveform of $\mathscr{E}_x^+$ for $t = t_0, t_1, \ldots, t_n, t_{n+1}$. To follow the point z_p for different values of t, we must maintain constant the amplitude of the last cosine term in (3-33a). This is accomplished by keeping its argument $\omega t - \beta_z z_p$ constant, that is,

$$\omega t - \beta_z z_p = C_0 = \text{constant} \tag{3-34}$$

which when differentiated with respect to time reduces to

$$\omega(1) - \beta_z \frac{dz_p}{dt} = 0 \Rightarrow \frac{dz_p}{dt} = v_p = +\frac{\omega}{\beta_z} \tag{3-35}$$

The point z_p is referred to as an *equiphase* point and its velocity is denoted as the *phase velocity*. A similar procedure can be used to demonstrate that the second complex exponential term in (3-31) represents a wave that travels in the $-z$ direction.

B. SOURCE-FREE AND LOSSY MEDIA

When the media in which the waves are traveling are lossy ($\sigma \neq 0$) but source-free ($\mathbf{J}_i = \mathbf{M}_i = q_{ve} = q_{vm} = 0$), the vector wave equations that the complex electric **E** and magnetic **H** field intensities must satisfy are (3-17a) and (3-17b). As for the lossless case, let us examine the solution to one of them; the solution to the other can then be written by inspection once the solution to the first has been obtained. We choose to consider the solution for the electric field intensity **E**, which must satisfy (3-17a). An extended presentation of electromagnetic wave propagation in lossy media can be found in [4].

In a rectangular coordinate system, the general solution for $\mathbf{E}(x, y, z)$ can be written as

$$\mathbf{E}(x, y, z) = \hat{a}_x E_x(x, y, z) + \hat{a}_y E_y(x, y, z) + \hat{a}_z E_z(x, y, z) \tag{3-36}$$

When (3-36) is substituted into (3-17a), we can write that

$$\nabla^2 \mathbf{E} - \gamma^2 \mathbf{E} = \nabla^2(\hat{a}_x E_x + \hat{a}_y E_y + \hat{a}_z E_z) - \gamma^2(\hat{a}_x E_x + \hat{a}_y E_y + \hat{a}_z E_z) = 0 \tag{3-37}$$

which reduces to three scalar wave equations of

$$\nabla^2 E_x(x, y, z) - \gamma^2 E_x(x, y, z) = 0 \tag{3-37a}$$

$$\nabla^2 E_y(x, y, z) - \gamma^2 E_y(x, y, z) = 0 \tag{3-37b}$$

$$\nabla^2 E_z(x, y, z) - \gamma^2 E_z(x, y, z) = 0 \tag{3-37c}$$

where

$$\gamma^2 = j\omega\mu(\sigma + j\omega\varepsilon) \tag{3-37d}$$

If we were to allow for positive and negative values of σ

$$\gamma = \pm\sqrt{j\omega\mu(\sigma + j\omega\varepsilon)} = \begin{cases} \pm(\alpha + j\beta) & \text{for } +\sigma \\ \pm(\alpha - j\beta) & \text{for } -\sigma \end{cases} \tag{3-37e}$$

In (3-37e),

γ = propagation constant

α = attenuation constant (Np/m)

β = phase constant (rad/m)

where α and β are assumed to be real and positive. Although some authors choose to represent the phase constant by k, the symbol β will be used throughout this book.

Examining (3-37e) reveals that there are four possible combinations for the form of γ. That is,

$$\gamma = \begin{cases} +(\alpha + j\beta) & \text{(3-38a)} \\ -(\alpha + j\beta) & \text{(3-38b)} \\ +(\alpha - j\beta) & \text{(3-38c)} \\ -(\alpha - j\beta) & \text{(3-38d)} \end{cases}$$

Of the four combinations, only one will be appropriate for our solution. That form will be selected once the solutions to any of (3-37a) through (3-37c) have been decided.

Since all three equations represented by (3-37a) through (3-37c) are of the same form, let us examine only one of them. We choose to work first with (3-37a) whose solution can be derived using the method of *separation of variables*. Using a similar procedure as for the lossless case, we can write that

$$E_x(x, y, z) = f(x)g(y)h(z) \tag{3-39}$$

where it can be shown that $f(x)$ has solutions of the form

$$f_1(x) = A_1 e^{-\gamma_x x} + B_1 e^{+\gamma_x x} \tag{3-40a}$$

or

$$f_2(x) = C_1 \cosh(\gamma_x x) + D_1 \sinh(\gamma_x x) \tag{3-40b}$$

and $g(y)$ can be expressed as

$$g_1(y) = A_2 e^{-\gamma_y y} + B_2 e^{+\gamma_y y} \tag{3-41a}$$

or

$$g_2(y) = C_2 \cosh(\gamma_y y) + D_2 \sinh(\gamma_y y) \tag{3-41b}$$

and $h(z)$ as

$$h_1(z) = A_3 e^{-\gamma_z z} + B_3 e^{+\gamma_z z} \tag{3-42a}$$

or

$$h_2(z) = C_3 \cosh(\gamma_z z) + D_3 \sinh(\gamma_z z) \tag{3-42b}$$

Whereas (3-40a) through (3-42b) are appropriate solutions for f, g, and h of (3-39), which satisfy (3-37a), the constraint equation takes the form of

$$\boxed{\gamma_x^2 + \gamma_y^2 + \gamma_z^2 = \gamma^2} \tag{3-43}$$

The appropriate forms of f, g, and h chosen to represent the solution of $E_x(x, y, z)$, as given by (3-39), must be made by examining the geometry of the problem in question. As for the lossless case, the exponentials represent attenuating traveling waves and the hyperbolic cosines and sines represent attenuating standing waves. These and other waves types are listed in Table 3-1.

To decide on the appropriate form for any of the γ's (whether it be γ_x, γ_y, γ_z, or γ), let us choose the form of γ_z by examining one of the exponentials in (3-42a). We choose to work with the first one. The four possible combinations for γ_z, according to (3-38a) through (3-38d) will be

$$\gamma_z = \begin{cases} +(\alpha_z + j\beta_z) & \text{(3-44a)} \\ -(\alpha_z + j\beta_z) & \text{(3-44b)} \\ +(\alpha_z - j\beta_z) & \text{(3-44c)} \\ -(\alpha_z - j\beta_z) & \text{(3-44d)} \end{cases}$$

If we want the first exponential in (3-42a) to represent a decaying wave which travels in the $+z$ direction, then by substituting (3-44a) through (3-44d) into it we can write that

$$h_1^+(z) = \begin{cases} A_3 e^{-\gamma_z z} = A_3 e^{-\alpha_z z} e^{-j\beta_z z} & \text{(3-45a)} \\ A_3 e^{-\gamma_z z} = A_3 e^{+\alpha_z z} e^{+j\beta_z z} & \text{(3-45b)} \\ A_3 e^{-\gamma_z z} = A_3 e^{-\alpha_z z} e^{+j\beta_z z} & \text{(3-45c)} \\ A_3 e^{-\gamma_z z} = A_3 e^{+\alpha_z z} e^{-j\beta_z z} & \text{(3-45d)} \end{cases}$$

By examining (3-45a) through (3-45d) and assuming $e^{j\omega t}$ time variations, the following statements can be made:

1. Equation 3-45a represents a wave that travels in the $+z$ direction, as determined by $e^{-j\beta_z z}$, and it decays in that direction, as determined by $e^{-\alpha_z z}$.
2. Equation 3-45b represents a wave that travels in the $-z$ direction, as determined by $e^{+j\beta_z z}$, and it decays in that direction, as determined by $e^{+\alpha_z z}$.
3. Equation 3-45c represents a wave that travels in the $-z$ direction, as determined by $e^{+j\beta_z z}$, and it is increasing in that direction, as determined by $e^{-\alpha_z z}$.
4. Equation 3-45d represents a wave that travels in the $+z$ direction, as determined by $e^{-j\beta_z z}$, and it is increasing in that direction, as determined by $e^{+\alpha_z z}$.

From the preceding statements it is apparent that for $e^{-\gamma_z z}$ to represent a wave that travels in the $+z$ direction and that concurrently also decays (to represent propagation in passive lossy media), and to satisfy the conservation of energy laws, the only correct form of γ_z is that of (3-44a). The same conclusion will result if the second exponential of (3-42a) represents a wave that travels in the $-z$ direction and that concurrently also decays. Thus the general form of any γ_i (whether it be γ_x, γ_y, γ_z, or γ), as given by (3-38a) through (3-38d), is

$$\gamma_i = \alpha_i + j\beta_i \tag{3-46}$$

Whereas the forms of f, g, and h [as given by (3-40a) through (3-42b)] are used to arrive at the solution for the complex form of E_x as given by (3-39), the instantaneous form of $\mathscr{E}_x$ can be obtained by using the relation of (3-32). A similar procedure can be used to derive the solutions of the other components of $\mathbf{E}$ (E_y and E_z), all those of $\mathbf{H}$ (H_x, H_y, and H_z), and of their instantaneous counterparts.

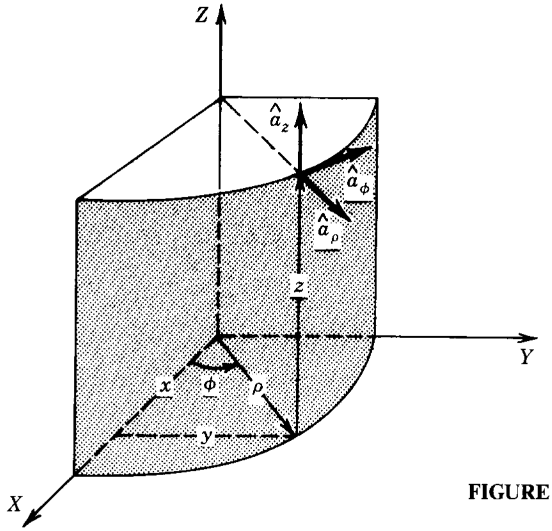

FIGURE 3-4 Cylindrical coordinate system and corresponding unit vectors.

3.4.2 Cylindrical Coordinate System

If the geometry of the system is of a cylindrical configuration, it would be very advisable to solve the boundary-value problem for the **E** and **H** fields using cylindrical coordinates. Maxwell's equations and the vector wave equations, which the **E** and **H** fields must satisfy, should be solved using cylindrical coordinates. Let us first consider the solution for **E** for a source-free and lossless medium. A similar procedure can be used for **H**. To maintain some simplicity in the mathematics, we will examine only lossless media.

In cylindrical coordinates a general solution to the vector wave equation for source-free and lossless media, as given by (3-18a), can be written as

$$\mathbf{E}(\rho, \phi, z) = \hat{a}_\rho E_\rho(\rho, \phi, z) + \hat{a}_\phi E_\phi(\rho, \phi, z) + \hat{a}_z E_z(\rho, \phi, z) \qquad (3\text{-}47)$$

where ρ, ϕ, and z are the cylindrical coordinates as illustrated in Figure 3-4. Substituting (3-47) into (3-18a), we can write that

$$\nabla^2(\hat{a}_\rho E_\rho + \hat{a}_\phi E_\phi + \hat{a}_z E_z) = -\beta^2(\hat{a}_\rho E_\rho + \hat{a}_\phi E_\phi + \hat{a}_z E_z) \qquad (3\text{-}48)$$

which does not reduce to three simple scalar wave equations, similar to those of (3-20a) through (3-20c) for (3-20), because

$$\nabla^2(\hat{a}_\rho E_\rho) \neq \hat{a}_\rho \nabla^2 E_\rho \qquad (3\text{-}49a)$$

$$\nabla^2(\hat{a}_\phi E_\phi) \neq \hat{a}_\phi \nabla^2 E_\phi \qquad (3\text{-}49b)$$

However, because

$$\nabla^2(\hat{a}_z E_z) = \hat{a}_z \nabla^2 E_z \qquad (3\text{-}49c)$$

one of the three scalar equations to which (3-48) reduces is

$$\nabla^2 E_z + \beta^2 E_z = 0 \qquad (3\text{-}50)$$

The other two are of more complex form and they will be addressed in what follows.

Before we derive the other two scalar equations [in addition to (3-50)] to which (3-48) reduces, let us attempt to give a physical explanation of (3-49a), (3-49b), and (3-49c). By examining two different points (ρ_1, ϕ_1, z_1) and (ρ_2, ϕ_2, z_2) and their corresponding unit vectors on a cylindrical surface (as shown in Figure 3-4), we see that the directions of $\hat{a}_\rho$ and $\hat{a}_\phi$ have changed from one point to another (they are not parallel) and therefore cannot be treated as constants but rather are functions of ρ, ϕ, and z. In contrast, the unit vector $\hat{a}_z$ at the two points is pointed in the same direction (is parallel). The same is true for the unit vectors $\hat{a}_x$ and $\hat{a}_y$ in Figure 3-1.

Let us now return to the solution of (3-48). Since (3-48) does not reduce to (3-49a) and (3-49b), although it does satisfy (3-49c), how do we solve (3-48)? The procedure that follows can be used to reduce (3-48) to three scalar partial differential equations.

The form of (3-48) written in general as

$$\nabla^2 \mathbf{E} = -\beta^2 \mathbf{E} \tag{3-51}$$

was placed in this form by utilizing the vector identity of (3-5) during its derivation. Generally we are under the impression that we do not know how to perform the Laplacian of a vector ($\nabla^2 \mathbf{E}$) as given by the left side of (3-51). However, by utilizing (3-5) we can rewrite the left side of (3-51) as

$$\nabla^2 \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla \times \nabla \times \mathbf{E} \tag{3-52}$$

whose terms can be expanded in any coordinate system. Using (3-52) we can write (3-51) as

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla \times \nabla \times \mathbf{E} = -\beta^2 \mathbf{E} \tag{3-53}$$

which is an alternate, but not as commonly recognizable, form of the vector wave equation for the electric field in source-free and lossless media.

Assuming a solution for the electric field of the form given by (3-47), we can expand (3-53) and reduce it to three scalar partial differential equations of the form

$$\nabla^2 E_\rho + \left(-\frac{E_\rho}{\rho^2} - \frac{2}{\rho^2} \frac{\partial E_\phi}{\partial \phi} \right) = -\beta^2 E_\rho \tag{3-54a}$$

$$\nabla^2 E_\phi + \left(-\frac{E_\phi}{\rho^2} + \frac{2}{\rho^2} \frac{\partial E_\rho}{\partial \phi} \right) = -\beta^2 E\phi \tag{3-54b}$$

$$\nabla^2 E_z = -\beta^2 E_z \tag{3-54c}$$

In each of (3-54a) through (3-54c) $\nabla^2 \psi(\rho, \phi, z)$ is the Laplacian of a scalar that in cylindrical coordinates takes the form of

$$\begin{aligned} \nabla^2 \psi(\rho, \phi, z) &= \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \psi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2} \\ &= \frac{\partial^2 \psi}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \psi}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2} \end{aligned} \tag{3-55}$$

Equations 3-54a and 3-54b are *coupled* (each contains more than one electric field component) second-order partial differential equations, which are the most difficult to solve. However, (3-54c) is an *uncoupled* second-order partial differential equation whose solution will be most useful in the construction of TE^z and TM^z mode solutions of boundary-value problems, as discussed in Chapters 6 and 9.

In expanded form (3-54c) can then be written as

$$\frac{\partial^2\psi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\psi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2} = -\beta^2\psi \tag{3-56}$$

where $\psi(\rho, \phi, z)$ is a scalar function that can represent a field or a vector potential component. Assuming a separable solution for $\psi(\rho, \phi, z)$ of the form

$$\psi(\rho, \phi, z) = f(\rho)g(\phi)h(z) \tag{3-57}$$

and substituting it into (3-56), we can write that

$$gh\frac{\partial^2 f}{\partial\rho^2} + gh\frac{1}{\rho}\frac{\partial f}{\partial\rho} + fh\frac{1}{\rho^2}\frac{\partial^2 g}{\partial\phi^2} + fg\frac{\partial^2 h}{\partial z^2} = -\beta^2 fgh \tag{3-58}$$

Dividing both sides of (3-58) by fgh and replacing the partials by ordinary derivatives reduces (3-58) to

$$\frac{1}{f}\frac{d^2 f}{d\rho^2} + \frac{1}{f}\frac{1}{\rho}\frac{df}{d\rho} + \frac{1}{g}\frac{1}{\rho^2}\frac{d^2 g}{d\phi^2} + \frac{1}{h}\frac{d^2 h}{dz^2} = -\beta^2 \tag{3-59}$$

The last term on the left side of (3-59) is only a function of z. Therefore, using the discussion of Section 3.4.1, we can write that

$$\frac{1}{h}\frac{d^2 h}{dz^2} = -\beta_z^2 \Rightarrow \frac{d^2 h}{dz^2} = -\beta_z^2 h \tag{3-60}$$

where β_z is a constant. Substituting (3-60) into (3-59) and multiplying both sides by ρ^2, reduces it to

$$\frac{\rho^2}{f}\frac{d^2 f}{d\rho^2} + \frac{\rho}{f}\frac{df}{d\rho} + \frac{1}{g}\frac{d^2 g}{d\phi^2} + \left(\beta^2 - \beta_z^2\right)\rho^2 = 0 \tag{3-61}$$

Since the third term on the left side of (3-61) is only a function of ϕ, it can be set equal to a constant $-m^2$. Thus we can write that

$$\frac{1}{g}\frac{d^2 g}{d\phi^2} = -m^2 \Rightarrow \frac{d^2 g}{d\phi^2} = -m^2 g \tag{3-62}$$

Letting

$$\beta^2 - \beta_z^2 = \beta_\rho^2 \Rightarrow \beta_\rho^2 + \beta_z^2 = \beta^2 \tag{3-63}$$

then using (3-62), and multiplying both sides of (3-61) by f, we can reduce (3-61) to

$$\rho^2\frac{d^2 f}{d\rho^2} + \rho\frac{df}{d\rho} + \left[\left(\beta_\rho\rho\right)^2 - m^2\right]f = 0 \tag{3-64}$$

Equation 3-63 is referred to as the constraint equation for the solution to the wave equation in cylindrical coordinates, and equation 3-64 is recognized as the classic *Bessel differential equation* [1–3, 5–10].

In summary then, the partial differential equation (3-56) whose solution was assumed to be separable of the form given by (3-57) reduces to the three differential equations 3-60, 3-62, 3-64 and the constraint equation 3-63. Thus

$$\nabla^2\psi(\rho,\phi,z) = \frac{\partial^2\psi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\psi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2} = -\beta^2\psi \qquad (3\text{-}65)$$

where

$$\psi(\rho,\phi,z) = f(\rho)g(\phi)h(z) \qquad (3\text{-}65a)$$

reduces to

$$\boxed{\rho^2\frac{d^2f}{d\rho^2} + \rho\frac{df}{d\rho} + \left[(\beta_\rho\rho)^2 - m^2\right]f = 0} \qquad (3\text{-}66a)$$

$$\boxed{\frac{d^2g}{d\phi^2} = -m^2 g} \qquad (3\text{-}66b)$$

$$\boxed{\frac{d^2h}{dz^2} = -\beta_z^2 h} \qquad (3\text{-}66c)$$

with

$$\boxed{\beta_\rho^2 + \beta_z^2 = \beta^2} \qquad (3\text{-}66d)$$

Solutions to (3-66a), (3-66b), and (3-66c) take the form, respectively, of

$$f_1(\rho) = A_1 J_m(\beta_\rho\rho) + B_1 Y_m(\beta_\rho\rho) \qquad (3\text{-}67a)$$

or

$$f_2(\rho) = C_1 H_m^{(1)}(\beta_\rho\rho) + D_1 H_m^{(2)}(\beta_\rho\rho) \qquad (3\text{-}67b)$$

and

$$g_1(\phi) = A_2 e^{-jm\phi} + B_2 e^{+jm\phi} \qquad (3\text{-}68a)$$

or

$$g_2(\phi) = C_2\cos(m\phi) + D_2\sin(m\phi) \qquad (3\text{-}68b)$$

and

$$h_1(z) = A_3 e^{-j\beta_z z} + B_3 e^{+j\beta_z z} \qquad (3\text{-}69a)$$

or

$$h_2(z) = C_3\cos(\beta_z z) + D_3\sin(\beta_z z) \qquad (3\text{-}69b)$$

In (3-67a) $J_m(\beta_\rho\rho)$ and $Y_m(\beta_\rho\rho)$ represent, respectively, the Bessel functions of the first and second kind; $H_m^{(1)}(\beta_\rho\rho)$ and $H_m^{(2)}(\beta_\rho\rho)$ in (3-67b) represent, respectively, the Hankel functions of the first and second kind. A more detailed discussion of Bessel and Hankel functions is found in Appendix IV.

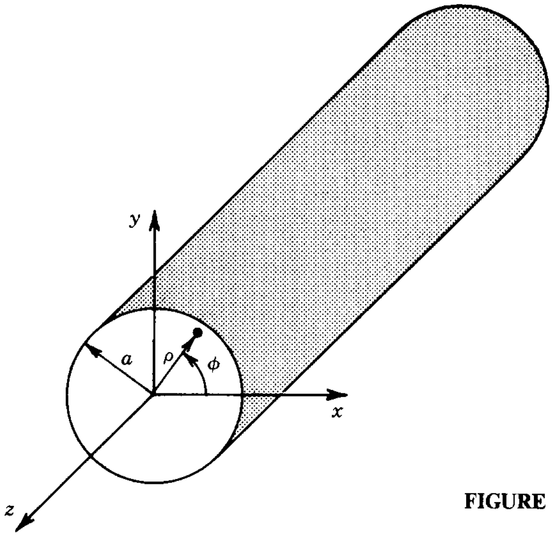

FIGURE 3-5 Cylindrical waveguide of the circular cross section.

Although (3-67a) through (3-69b) are valid solutions for $f(\rho)$, $g(\phi)$, and $h(z)$, the most appropriate form will depend on the problem in question. For example, for the cylindrical waveguide of Figure 3-5 the most convenient solutions for $f(\rho)$, $g(\phi)$, and $h(z)$ are those given, respectively, by (3-67a), (3-68b), and (3-69a). Thus we can write

$$\begin{aligned}\psi_1(\rho, \phi, z) &= f(\rho)g(\phi)h(z)\\ &= \left[A_1 J_m(\beta_\rho \rho) + B_1 Y_m(\beta_\rho \rho)\right]\\ &\quad \times \left[C_2 \cos(m\phi) + D_2 \sin(m\phi)\right]\left[A_3 e^{-j\beta_z z} + B_3 e^{+j\beta_z z}\right] \qquad (3\text{-}70)\end{aligned}$$

These forms for $f(\rho)$, $g(\phi)$, and $h(z)$ were chosen in cylindrical coordinates for the following reasons.

1. Bessel functions of (3-67a) are used to represent standing waves whereas Hankel functions of (3-67b) represent traveling waves.
2. Exponentials of (3-68a) represent traveling waves whereas the cosines and sines of (3-68b) represent periodic waves.
3. Exponentials of (3-69a) represent traveling waves whereas the cosines and sines of (3-69b) represent standing waves.

Wave functions representing various radial waves in cylindrical coordinates are found listed in Table 3-2.

Within the circular waveguide of Figure 3-5 standing waves are created in the radial (ρ) direction, periodic waves in the phi (ϕ) direction, and traveling waves in the z direction. For the fields to be finite at $\rho = 0$, where $Y_m(\beta_\rho \rho)$ possesses a singularity, (3-70) reduces to

$$\psi_1(\rho, \phi, z) = A_1 J_m(\beta_\rho \rho)\left[C_2 \cos(m\phi) + D_2 \sin(m\phi)\right]\left[A_3 e^{-j\beta_z z} + B_3 e^{+j\beta_z z}\right] \qquad (3\text{-}70a)$$

TABLE 3-2
Wave functions, zeroes, and infinities for radial wave functions in cylindrical coordinates

Wave type	Wave functions	Zeroes of wave functions	Infinities of wave functions
Traveling waves	$H_m^{(1)}(\beta\rho) = J_m(\beta\rho) + jY_m(\beta\rho)$ for $-\rho$ travel	$\beta\rho \to +j\infty$	$\beta\rho = 0$ $\beta\rho \to -j\infty$
	$H_m^{(2)}(\beta\rho) = J_m(\beta\rho) - jY_m(\beta\rho)$ for $+\rho$ travel	$\beta\rho \to -j\infty$	$\beta\rho = 0$ $\beta\rho \to +j\infty$
Standing waves	$J_m(\beta\rho)$ for $\pm\rho$	Infinite number (see Table 9-2)	$\beta\rho \to \pm j\infty$
	$Y_m(\beta\rho)$ for $\pm\rho$	Infinite number	$\beta\rho = 0$ $\beta\rho \to \pm j\infty$
Evanescent waves	$K_m(\alpha\rho) = \frac{\pi}{2}(-j)^{m+1}H_m^{(2)}(-j\alpha\rho)$ for $+\rho$	$\alpha\rho \to +\infty$	
	$I_m(\alpha\rho) = j^m J_m(-j\alpha\rho)$ for $-\rho$		$\alpha\rho \to +\infty$ for integer orders
Attenuating traveling waves	$H_m^{(1)}(\gamma\rho) = H_m^{(1)}(\alpha\rho + j\beta\rho)$ for $-\rho$ travel	$\gamma\rho \to +j\infty$	$\gamma\rho \to -j\infty$
	$H_m^{(2)}(\gamma\rho) = H_m^{(2)}(\alpha\rho + j\beta\rho)$ for $+\rho$ travel	$\gamma\rho \to -j\infty$	$\gamma\rho \to +j\infty$
Attenuating standing waves	$J_m(\gamma\rho) = J_m(\alpha\rho + j\beta\rho)$ for $\pm\rho$	Infinite number	$\gamma\rho \to \pm j\infty$
	$Y_m(\gamma\rho) = Y_m(\alpha\rho + j\beta\rho)$ for $\pm\rho$	Infinite number	$\gamma\rho \to \pm j\infty$

To represent the fields in the region outside the cylinder, a typical solution for $\psi(\rho, \phi, z)$ would take the form of

$$\psi_2(\rho, \phi, z) = B_1 H_m^{(2)}(\beta_\rho \rho)[C_2 \cos(m\phi) + D_2 \sin(m\phi)][A_3 e^{-j\beta_z z} + B_3 e^{+j\beta_z z}] \tag{3-70b}$$

whereby the Hankel function of the second kind $H_m^{(2)}(\beta_\rho \rho)$ has replaced the Bessel function of the first kind $J_m(\beta_\rho \rho)$ because outward traveling waves are formed outside the cylinder, in contrast to the standing waves inside the cylinder.

More details concerning the application and properties of Bessel and Hankel function can be found in Chapter 9.

3.4.3 Spherical Coordinate System

Spherical coordinates should be utilized in solving problems that exhibit spherical geometries. As for the rectangular and cylindrical geometries, the electric and magnetic fields of a spherical geometry boundary-value problem must satisfy the corresponding vector wave equation, which is most conveniently solved in spherical coordinates as illustrated in Figure 3-6.

To simplify the problem, let us assume that the space in which the electric and magnetic fields must be solved is source-free and lossless. A general solution for the

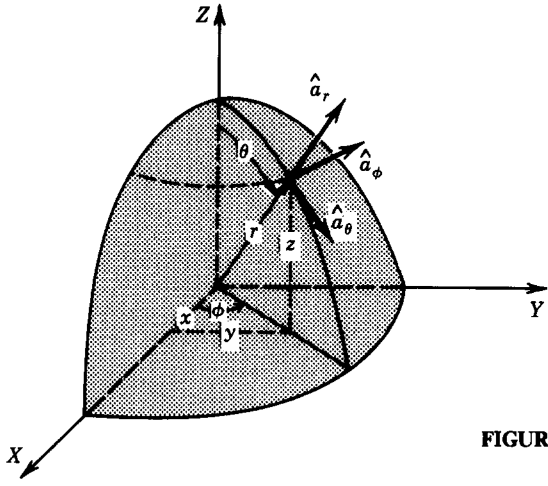

FIGURE 3-6 Spherical coordinate system and corresponding unit vectors.

electric field can then be written as

$$\mathbf{E}(r,\theta,\phi) = \hat{a}_r E_r(r,\theta,\phi) + \hat{a}_\theta E_\theta(r,\theta,\phi) + \hat{a}_\phi E_\phi(r,\theta,\phi) \quad (3\text{-}71)$$

Substituting (3-71) into the vector wave equation of (3-18a), we can write that

$$\nabla^2(\hat{a}_r E_r + \hat{a}_\theta E_\theta + \hat{a}_\phi E_\phi) = -\beta^2(\hat{a}_r E_r + \hat{a}_\theta E_\theta + \hat{a}_\phi E_\phi) \quad (3\text{-}72)$$

Since

$$\nabla^2(\hat{a}_r E_r) \neq \hat{a}_r \nabla^2 E_r \quad (3\text{-}73a)$$

$$\nabla^2(\hat{a}_\theta E_\theta) \neq \hat{a}_\theta \nabla^2 E_\theta \quad (3\text{-}73b)$$

$$\nabla^2(\hat{a}_\phi E_\phi) \neq \hat{a}_\phi \nabla^2 E_\phi \quad (3\text{-}73c)$$

(3-72) does not reduce to three simple scalar wave equations, similar to those of (3-20a) through (3-20c) for (3-20). Therefore the reduction of (3-72) to three scalar partial differential equations must proceed in a different manner. In fact, the method used here will be similar to that utilized in cylindrical coordinates to reduce the vector wave equation to three scalar partial differential equations.

To accomplish this, we first rewrite the vector wave equation of (3-51) in a form given by (3-53) where now all the operators on the left side can be performed in any coordinate system. Substituting (3-71) into (3-53) shows that after some lengthy mathematical manipulations (3-53) reduces to three scalar partial differential equations of the form

$$\nabla^2 E_r - \frac{2}{r^2}\left(E_r + E_\theta \cot\theta + \csc\theta \frac{\partial E_\phi}{\partial \phi} + \frac{\partial E_\theta}{\partial \theta}\right) = -\beta^2 E_r \quad (3\text{-}74a)$$

$$\nabla^2 E_\theta - \frac{1}{r^2}\left(E_\theta \csc^2\theta - 2\frac{\partial E_r}{\partial \theta} + 2\cot\theta\csc\theta \frac{\partial E_\phi}{\partial \phi}\right) = -\beta^2 E_\theta \quad (3\text{-}74b)$$

$$\nabla^2 E_\phi - \frac{1}{r^2}\left(E_\phi \csc^2\theta - 2\csc\theta \frac{\partial E_r}{\partial \phi} - 2\cot\theta\csc\theta \frac{\partial E_\theta}{\partial \phi}\right) = -\beta^2 E_\phi \quad (3\text{-}74c)$$

Unfortunately, all three of the preceding partial differential equations are coupled. This means each contains more than one component of the electric field and would be most difficult to solve in its present form. However, as will be shown in Chapter 10, TE^r and TM^r wave mode solutions can be formed that in spherical coordinates must satisfy the scalar wave equation of

$$\nabla^2\psi(r,\theta,\phi) = -\beta^2\psi(r,\theta,\phi) \tag{3-75}$$

where $\psi(r, \theta, \phi)$ is a scalar function that can represent a field or a vector potential component. Therefore it would be advisable here to demonstrate the solution to (3-75) in spherical coordinates.

Assuming a separable solution for $\psi(r, \theta, \phi)$ of the form

$$\psi(r,\theta,\phi) = f(r)g(\theta)h(\phi) \tag{3-76}$$

we can write the expanded form of (3-75)

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left\{r^2\frac{\partial\psi}{\partial r}\right\} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left\{\sin\theta\frac{\partial\psi}{\partial\theta}\right\} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} = -\beta^2\psi \tag{3-77}$$

as

$$gh\frac{1}{r^2}\frac{\partial}{\partial r}\left\{r^2\frac{\partial f}{\partial r}\right\} + fh\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left\{\sin\theta\frac{\partial g}{\partial\theta}\right\} + fg\frac{1}{r^2\sin^2\theta}\frac{\partial^2 h}{\partial\phi^2} = -\beta^2 fgh \tag{3-78}$$

Dividing both sides by fgh, multiplying by $r^2\sin^2\theta$, and replacing the partials by ordinary derivatives reduces (3-78) to

$$\frac{\sin^2\theta}{f}\frac{d}{dr}\left\{r^2\frac{df}{dr}\right\} + \frac{\sin\theta}{g}\frac{d}{d\theta}\left\{\sin\theta\frac{dg}{d\theta}\right\} + \frac{1}{h}\frac{d^2h}{d\phi^2} = -(\beta r\sin\theta)^2 \tag{3-79}$$

Since the last term on the left side of (3-79) is only a function of ϕ, it can be set equal to

$$\frac{1}{h}\frac{d^2h}{d\phi^2} = -m^2 \Rightarrow \frac{d^2h}{d\phi^2} = -m^2h \tag{3-80}$$

where m is a constant.

Substituting (3-80) into (3-79), dividing both sides by $\sin^2\theta$, and transposing the term from the right to the left side reduces (3-79) to

$$\frac{1}{f}\frac{d}{dr}\left\{r^2\frac{df}{dr}\right\} + (\beta r)^2 + \frac{1}{g\sin\theta}\frac{d}{d\theta}\left\{\sin\theta\frac{dg}{d\theta}\right\} - \left\{\frac{m}{\sin\theta}\right\}^2 = 0 \tag{3-81}$$

Since the last two terms on the left side of (3-81) are only a function of θ, we can set them equal to

$$\frac{1}{g\sin\theta}\frac{d}{d\theta}\left\{\sin\theta\frac{dg}{d\theta}\right\} - \left\{\frac{m}{\sin\theta}\right\}^2 = -n(n+1) \tag{3-82}$$

where n is usually an integer. Equation 3-82 is closely related to the well-known *Legendre differential equation* (see Appendix V) [1–3, 6–10].

Substituting (3-82) into (3-81) reduces it to

$$\frac{1}{f}\frac{d}{dr}\left(r^2\frac{df}{dr}\right) + (\beta r)^2 - n(n+1) = 0 \qquad (3\text{-}83)$$

which is closely related to the Bessel differential equation (see Appendix IV).

In summary then, the scalar wave equation 3-75 whose expanded form in spherical coordinates can be written as

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} = -\beta^2\psi \qquad (3\text{-}84)$$

and whose separable solution takes the form of

$$\boxed{\psi(r,\theta,\phi) = f(r)g(\theta)h(\phi)} \qquad (3\text{-}85)$$

reduces to the three scalar differential equations

$$\boxed{\frac{d}{dr}\left(r^2\frac{df}{dr}\right) + \left[(\beta r)^2 - n(n+1)\right]f = 0} \qquad (3\text{-}86a)$$

$$\boxed{\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dg}{d\theta}\right) + \left[n(n+1) - \left(\frac{m}{\sin\theta}\right)^2\right]g = 0} \qquad (3\text{-}86b)$$

$$\boxed{\frac{d^2h}{d\phi^2} = -m^2h} \qquad (3\text{-}86c)$$

where m and n are constants (usually integers).

Solutions to (3-86a) through (3-86c) take the forms, respectively, of

$$f_1(r) = A_1 j_n(\beta r) + B_1 y_n(\beta r) \qquad (3\text{-}87a)$$

or

$$f_2(r) = C_1 h_n^{(1)}(\beta r) + D_1 h_n^{(2)}(\beta r) \qquad (3\text{-}87b)$$

and

$$g_1(\theta) = A_2 P_n^m(\cos\theta) + B_2 P_n^m(-\cos\theta) \qquad n \neq \text{integer} \qquad (3\text{-}88a)$$

or

$$g_2(\theta) = C_2 P_n^m(\cos\theta) + D_2 Q_n^m(\cos\theta) \qquad n = \text{integer} \qquad (3\text{-}88b)$$

and

$$h_1(\phi) = A_3 e^{-jm\phi} + B_3 e^{+jm\phi} \qquad (3\text{-}89a)$$

or

$$h_2(\phi) = C_3\cos(m\phi) + D_3\sin(m\phi) \qquad (3\text{-}89b)$$

TABLE 3-3
Wave functions, zeroes, and infinites for radial waves in spherical coordinates

Wave type	Wave functions	Zeroes of wave functions	Infinities of wave functions
Traveling waves	$h_n^{(1)}(\beta r) = j_n(\beta r) + jy_n(\beta r)$ for $-r$ travel	$\beta r \to +j\infty$	$\beta r = 0$ $\beta r \to -j\infty$
	$h_n^{(2)}(\beta r) = j_n(\beta r) - jy_n(\beta r)$ for $+r$ travel	$\beta r \to -j\infty$	$\beta r = 0$ $\beta r \to +j\infty$
Standing waves	$j_n(\beta r)$ for $\pm r$	Infinite number	$\beta r \to \pm j\infty$
	$y_n(\beta r)$ for $\pm r$	Infinite number	$\beta r = 0$ $\beta r \to \pm j\infty$

In (3-87a) $j_n(\beta r)$ and $y_n(\beta r)$ are referred to, respectively, as the *spherical Bessel functions* of the first and second kind. They are used to represent radial standing waves, and they are related, respectively, to the corresponding regular Bessel functions $J_{n+1/2}(\beta r)$ and $Y_{n+1/2}(\beta r)$ by

$$j_n(\beta r) = \sqrt{\frac{\pi}{2\beta r}}\, J_{n+1/2}(\beta r) \tag{3-90a}$$

$$y_n(\beta r) = \sqrt{\frac{\pi}{2\beta r}}\, Y_{n+1/2}(\beta r) \tag{3-90b}$$

In (3-87b) $h_n^{(1)}(\beta r)$ and $h_n^{(2)}(\beta r)$ are referred to, respectively, as the *spherical Hankel functions* of the first and second kind. They are used to represent radial traveling waves, and they are related, respectively, to the regular Hankel functions $H_{n+1/2}^{(1)}(\beta r)$ and $H_{n+1/2}^{(2)}(\beta r)$ by

$$h_n^{(1)}(\beta r) = \sqrt{\frac{\pi}{2\beta r}}\, H_{n+1/2}^{(1)}(\beta r) \tag{3-91a}$$

$$h_n^{(2)}(\beta r) = \sqrt{\frac{\pi}{2\beta r}}\, H_{n+1/2}^{(2)}(\beta r) \tag{3-91b}$$

Wave functions used to represent radial traveling and standing waves in spherical coordinates are listed in Table 3-3. More details on the spherical Bessel and Hankel functions can be found in Appendix IV.

In (3-88a) and (3-88b) $P_n^m(\cos\theta)$ and $Q_n^m(\cos\theta)$ are referred to, respectively, as the *associated Legendre functions* of the first and second kind (more details can be found in Appendix V).

The appropriate solution forms of f, g, and h will depend on the problem in question. For example, a typical solution for $\psi(r,\theta,\phi)$ of (3-85) to represent the fields within a sphere as shown in Figure 3-7 may take the form

$$\begin{aligned}\psi_1(r,\theta,\phi) &= [A_1 j_n(\beta r) + B_1 y_n(\beta r)] \\ &\quad \times [C_2 P_n^m(\cos\theta) + D_2 Q_n^m(\cos\theta)][C_3\cos(m\phi) + D_3\sin(m\phi)]\end{aligned} \tag{3-92}$$

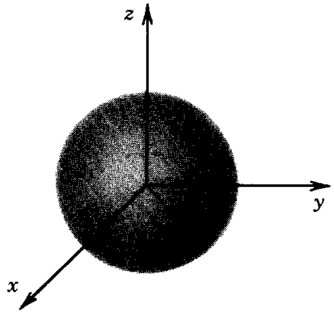

FIGURE 3-7 Geometry of a sphere of radius a.

For the fields to be finite at $r = 0$, where $y_n(\beta r)$ possesses a singularity, and for any value of θ, including $\theta = 0, \pi$ where $Q_n^m(\cos\theta)$ possesses singularities, (3-92) reduces to

$$\psi_1(r,\theta,\phi) = A_{mn} j_n(\beta r) P_n^m(\cos\theta)[C_3 \cos(m\phi) + D_3 \sin(m\phi)] \quad \text{(3-92a)}$$

To represent the fields outside a sphere a typical solution for $\psi(r, \theta, \phi)$ would take the form of

$$\psi_2(r,\theta,\phi) = B_{mn} h_n^{(2)}(\beta r) P_n^m(\cos\theta)[C_3 \cos(m\phi) + D_3 \sin(m\phi)] \quad \text{(3-92b)}$$

whereby the spherical Hankel function of the second kind $h_n^{(2)}(\beta r)$ has replaced the spherical Bessel function of the first kind $j_n(\beta r)$ because outward traveling waves are formed outside the sphere, in contrast to the standing waves inside the sphere.

Other spherical Bessel and Hankel functions that are most often encountered in boundary-value electromagnetic problems are those utilized by Schelkunoff [3, 11]. These spherical Bessel and Hankel functions, denoted in general by $\hat{B}_n(\beta r)$ to represent any of them, must satisfy the differential equation

$$\frac{d^2\hat{B}_n}{dr^2} + \left[\beta^2 - \frac{n(n+1)}{r^2}\right]\hat{B}_n = 0 \quad \text{(3-93)}$$

The spherical Bessel and Hankel functions that are solutions to this equation are related to other spherical Bessel and Hankel functions of (3-90a) through (3-91b), denoted here by $b_n(\beta r)$, and to the regular Bessel and Hankel functions, denoted here by $B_{n+1/2}(\beta r)$, by

$$\hat{B}_n(\beta r) = \beta r\, b_n(\beta r) = \beta r \sqrt{\frac{\pi}{2\beta r}}\, B_{n+1/2}(\beta r) = \sqrt{\frac{\pi\beta r}{2}}\, B_{n+1/2}(\beta r) \quad \text{(3-94)}$$

More details concerning the application and properties of the spherical Bessel and Hankel functions can be found in Chapter 10.

REFERENCES

1. F. B. Hilderbrand, *Advanced Calculus for Applications*, Prentice-Hall, Englewood Cliffs, N.J., 1962.
2. C. R. Wylie, Jr., *Advanced Engineering Mathematics*, McGraw-Hill, New York, 1960.

3. R. F. Harrington, *Time-Harmonic Electromagnetic Fields*, McGraw-Hill, New York, 1961.
4. R. B. Adler, L. J. Chu, and R. M. Fano, *Electromagnetic Energy Transmission and Radiation*, Chapter 8, Wiley, New York, 1960.
5. G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge Univ. Press, London, 1948.
6. W. R. Smythe, *Static and Dynamic Electricity*, McGraw-Hill, New York, 1941.
7. J. A. Stratton, *Electromagnetic Theory*, McGraw-Hill, New York, 1960.
8. P. M. Morse and H. Feshbach, *Methods of Theoretical Physics*, Parts I and II, McGraw-Hill, New York, 1953.
9. M. Abramowitz and I. A. Stegun (eds.), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards Applied Mathematics Series-55, U.S. Gov. Printing Office, Washington, D.C., 1966.
10. M. R. Spiegel, *Mathematical Handbook of Formulas and Tables*, Schaum's Outline Series, McGraw-Hill, New York, 1968.
11. S. A. Schelkunoff, *Electromagnetic Waves*, Van Nostrand, Princeton, N.J., 1943.

PROBLEMS

3.1. Derive the vector wave equations 3-16a and 3-16b for time-harmonic fields using the Maxwell equations of Table 1-4 for time-harmonic fields.

3.2. Verify that (3-28a) and (3-28b) are solutions to (3-26a).

3.3. Show that the second complex exponential in (3-31) represents a wave traveling in the $-z$ direction. Determine its phase velocity.

3.4. Using the method of separation of variables show that a solution to (3-37a) of the form (3-39) can be represented by (3-40a) through (3-43).

3.5. Show that the vector wave equation of (3-53) reduces, when **E** has a solution of the form (3-47), to the three scalar wave equations 3-54a through 3-54c.

3.6. Reduce (3-51) to (3-54a) through (3-54c) by expanding $\nabla^2\mathbf{E}$. Do not use (3-52); rather use the scalar Laplacian in cylindrical coordinates and treat **E** as a vector given by (3-47). Use that

$$\frac{\partial \hat{a}_\rho}{\partial \rho} = \frac{\partial \hat{a}_\phi}{\partial \rho} = \frac{\partial \hat{a}_z}{\partial \rho} = 0 = \frac{\partial \hat{a}_z}{\partial \phi} = \frac{\partial \hat{a}_\rho}{\partial z} = \frac{\partial \hat{a}_\phi}{\partial z} = \frac{\partial \hat{a}_z}{\partial z}$$

$$\frac{\partial \hat{a}_\rho}{\partial \phi} = \hat{a}_\phi \qquad \frac{\partial \hat{a}_\phi}{\partial \phi} = -\hat{a}_\rho$$

3.7. Using large argument asymptotic forms, show that Bessel and Hankel functions represent, respectively, standing and traveling waves in the radial direction.

3.8. Using large argument asymptotic forms and assuming $e^{j\omega t}$ time convention, show that Hankel functions of the first kind represent traveling waves in the $-\rho$ direction whereas Hankel functions of the second kind represent traveling waves in the $+\rho$ direction. The opposite would be true were the time variations of the $e^{-j\omega t}$ form.

3.9. Using large argument asymptotic forms show that Bessel functions of complex argument represent attenuating standing waves.

3.10. Assuming time variations of $e^{j\omega t}$ and using large argument asymptotic forms, show that Hankel functions of the first and second kind with complex arguments represent, respectively, attenuating traveling waves in the $-\rho$ and $+\rho$ directions.

3.11. Show that when **E** has a solution of the form 3-71, the vector wave equation 3-53 reduces to the three scalar wave equations 3-74a through 3-74c.

3.12. Reduce (3-51) to (3-74a) through (3-74c) by expanding $\nabla^2\mathbf{E}$. Do not use (3-52); rather use the scalar Laplacian in spherical coordinates and treat **E** as a vector given by (3-71). Use that

$$\frac{\partial \hat{a}_r}{\partial r} = \frac{\partial \hat{a}_\theta}{\partial r} = \frac{\partial \hat{a}_\phi}{\partial r} = 0$$

$$\frac{\partial \hat{a}_r}{\partial \theta} = \hat{a}_\theta \qquad \frac{\partial \hat{a}_\theta}{\partial \theta} = -\hat{a}_r \qquad \frac{\partial \hat{a}_\phi}{\partial \theta} = 0$$

$$\frac{\partial \hat{a}_r}{\partial \phi} = \sin\theta\hat{a}_\phi \qquad \frac{\partial \hat{a}_\theta}{\partial \phi} = \cos\theta\hat{a}_\phi \qquad \frac{\partial \hat{a}_\phi}{\partial \phi} = -\sin\theta\hat{a}_r - \cos\theta\hat{a}_\theta$$

3.13. Using large argument asymptotic forms show that spherical Bessel functions represent standing waves in the radial direction.

3.14. Show that spherical Hankel functions of the first and second kind represent, respectively, radial traveling waves in the $-r$ and $+r$ directions. Assume time variations of $e^{j\omega t}$ and large argument asymptotic expansions for the spherical Hankel functions.

3.15. Justify that associated Legendre functions represent standing waves in the θ direction of the spherical coordinate system.

3.16. Verify the relation 3-94 between the various forms of the spherical Bessel and Hankel functions and the regular Bessel and Hankel functions.