Radiation from Rectangular Aperture :

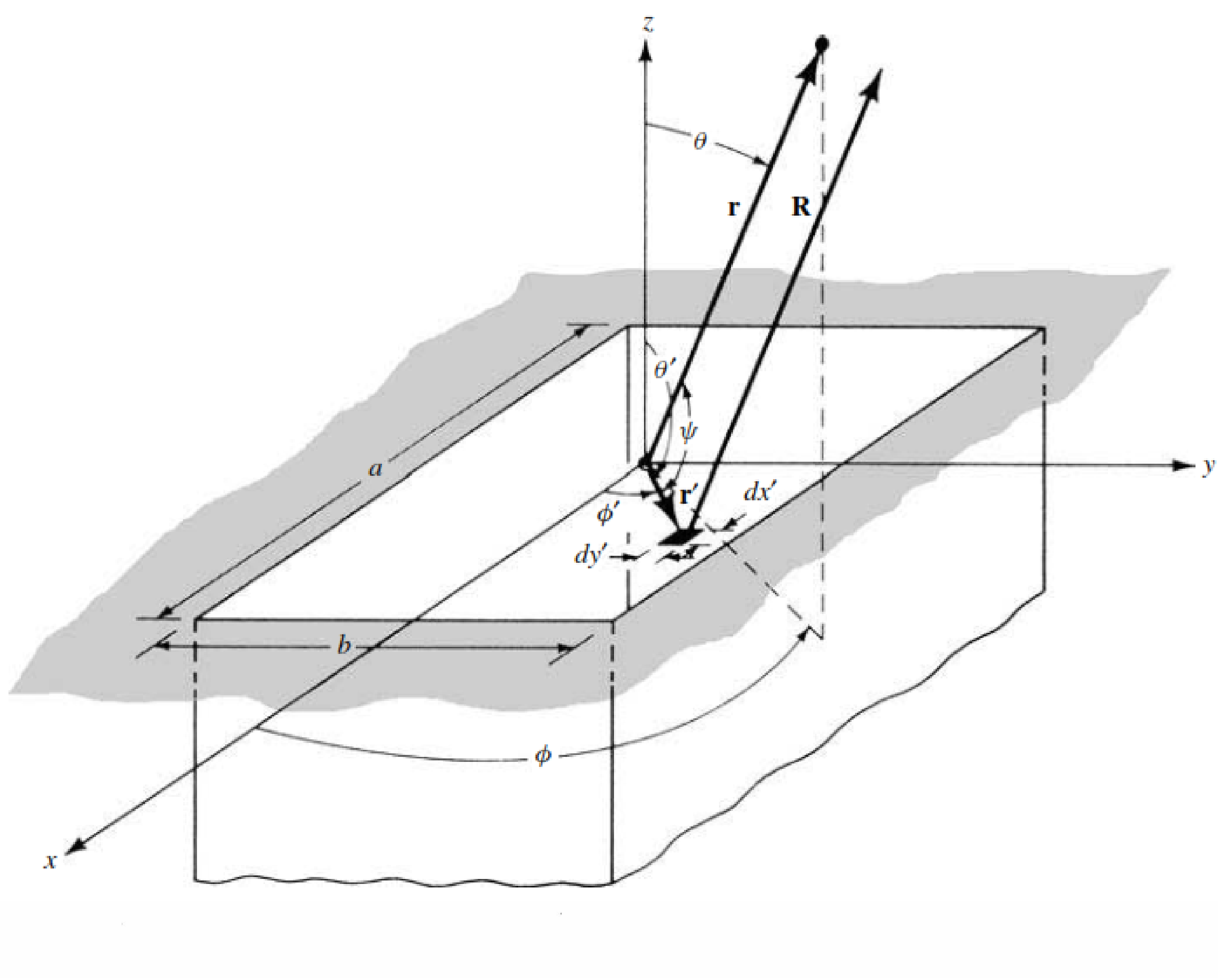
Uniform Distribution on an infinite Ground Plane :-

Consider an aperture of size $a \times b$ in the $z=0$ plane. For simplicity, assume that the aperture fields E_a and H_a are constant over the aperture and further,

$$\left.\begin{aligned} E_a &= a_y E_0 \\ \text{and} \quad H_a &= -a_x H_0 \end{aligned}\right\} \longrightarrow (1)$$

We also make an assumption that E_a and H_a fields are related via the free space Impedance,

$$\frac{|E_a|}{|H_a|} = \eta \longrightarrow (2)$$



Applying the field equivalence principle, the fields in the $z>0$ space can be computed from the equivalent current sheet.

$$\left.\begin{aligned} J_s &= a_z \times (-a_x H_0) = -a_y H_0 \\ \text{and } M_s &= -(a_z \times a_y E_0) = a_x E_0 \end{aligned}\right\} \rightarrow (3)$$

For convenience of integration, we use rectangular coordinates (x', y') for the source point and spherical coordinates (r, θ, ϕ) for the field point.

Using the far field approximations,

$$\left.\begin{aligned} R &\simeq r - r'\cos\psi \quad ; \quad \text{for phase variation} \\ R &\simeq r \quad ; \quad \text{for amplitude variation} \end{aligned}\right\} \rightarrow (4)$$

where ψ is the angle between the vector r and r'.

The expression of vector potential interms of current density J is given as,

$$A = \frac{\mu}{4\pi} \iint_S J \cdot \frac{e^{-jkR}}{R} \cdot ds' \rightarrow (5)$$

$$A = \frac{\mu}{4\pi} \iint_S J(x', y') \cdot \frac{e^{-jk(r - r'\cos\psi)}}{r} \cdot dx' dy'$$

$$A = \frac{\mu \cdot e^{-jkr}}{4\pi r} \iint_S J(x', y') \cdot e^{jkr'\cos\psi} \cdot dx' dy' \rightarrow (6)$$

$$F = \frac{\varepsilon}{4\pi r} e^{-jkr} \iint_S M(x', y')\, e^{jkr'\cos\psi}\, dx'dy' \rightarrow (7)$$

using the spherical co-ordinates the term in the exponent is written as,

$$r'\cos\psi = r' a_r$$

$$= (a_x x' + a_y y') \cdot (a_x \sin\theta \cos\phi + a_y \sin\theta \sin\phi)$$

$$= x' \sin\theta \cos\phi + y' \sin\theta \sin\phi \rightarrow (8)$$

substituting $ds' = dx'dy'$, $J_s = -a_y H_0$ and $M_s = a_x E_0$, we see that A has only y-component and F has only an x-component.

$$\therefore A_y = -\frac{\mu H_0}{4\pi r} e^{-jkr} \iint_S e^{jk(x'\sin\theta\cos\phi + y'\sin\theta\sin\phi)} \cdot dx'dy'$$

$$A_y = -\mu H_0 \cdot G \cdot I \longrightarrow (9)$$

similarly,

$$F_x = \varepsilon E_0 \cdot \frac{e^{-jkr}}{4\pi r} \iint_S e^{jk(x'\sin\theta\cos\phi + y'\sin\theta\sin\phi)} \cdot dx'dy'$$

$$F_x = \varepsilon E_0 \cdot G \cdot I \rightarrow (10)$$

where,

$$G = \frac{e^{-jkr}}{4\pi r} \rightarrow (11)$$

and $$I = \iint_S e^{jk(x' \sin\theta\cos\phi + y'\sin\theta\sin\phi)} \cdot dx'dy' \rightarrow (12)$$

using the integral,

$$\int_{-T/2}^{T/2} e^{j\alpha z} \cdot dz = T \cdot \left[\frac{\sin\left(\frac{\alpha}{2} T\right)}{\left(\frac{\alpha}{2} \cdot T\right)} \right] \rightarrow (13)$$

equ. (12) reduces to

$$I = \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} e^{jk(x'\sin\theta\cos\phi)} \cdot e^{jk(y'\sin\theta\sin\phi)} \cdot dx'dy'$$

$$I = \int_{-b/2}^{b/2} e^{jk\sin\theta\sin\phi\, y'} \cdot dy' \cdot \int_{-a/2}^{a/2} e^{jk\sin\theta\cos\phi\, x'} dx'$$

$$I = b \cdot \left[\frac{\sin\left(\frac{k\sin\theta\sin\phi}{2} \cdot b\right)}{\left(\frac{k\sin\theta\sin\phi}{2} \cdot b\right)} \right] a \cdot \left[\frac{\sin\left(\frac{k\sin\theta\cos\phi}{2} \cdot a\right)}{\left(\frac{k\sin\theta\cos\phi}{2} \cdot a\right)} \right] \rightarrow (14)$$

$$\left.\begin{aligned} \text{Let } X &= \frac{ka}{2} \sin\theta\cos\phi \\ \text{and } Y &= \frac{kb}{2} \sin\theta\sin\phi \end{aligned}\right\} \rightarrow (15)$$

$$\therefore I = ab \left(\frac{\sin X}{X}\right)\left(\frac{\sin Y}{Y}\right) \rightarrow (16)$$

The rectangular coordinate representation of the vector potentials A and F at any far field point $P(r, \theta, \phi)$ is split into components along a_r, a_θ and a_ϕ directions.

Using the rectangular to spherical coordinate transformation

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \rightarrow (17)$$

$$A_\theta = A_y \cos\theta \sin\phi \rightarrow (18)$$

$$A_\phi = A_y \cos\phi \rightarrow (19)$$

$$F_\theta = F_x \cos\theta \cos\phi \rightarrow (20)$$

$$F_\phi = -F_x \sin\phi \rightarrow (21)$$

In the far field, only θ and ϕ components of E and H fields are dominant. Although the radial components are not necessarily zero, they are negligible compared to the θ and ϕ components

The equations relating E and A, F are,

$$E_\theta = -j\omega (A_\theta + \eta F_\phi) \rightarrow (22)$$

$$E_\phi = -j\omega (A_\phi - \eta F_\theta) \rightarrow (23)$$

$$H_\theta = -\frac{E_\phi}{\eta} \rightarrow (24)$$

$$H_\phi = \frac{E_\theta}{\eta} \rightarrow (25)$$

substitute equ. (18) and (21) in (22)

$$E_\theta = -j\omega\left(A_y \cos\theta \sin\phi - \eta F_x \sin\phi\right)$$

$$= -j\omega\left(-\mu H_0 G I \cos\theta \sin\phi - \eta \xi E_0 G I \sin\phi\right)$$

$$= j\omega G I \sin\phi \left(\mu H_0 \cos\theta + \eta \xi E_0\right)$$

$$= j\omega G I \sin\phi \left(\mu \cdot \frac{E_0}{\eta} \cos\theta + \eta \xi E_0\right)$$

$$= j\omega G I E_0 \sin\phi \left(\frac{\mu}{\eta} \cos\theta + \eta\xi\right) \qquad \because \eta\xi = \frac{\mu}{\eta}$$

$$E_\theta = j\omega\eta\xi E_0 G I \sin\phi \left(1 + \cos\theta\right)$$

$$E_\theta = j\omega\eta\xi E_0 \cdot \frac{e^{-jkr}}{4\pi r} ab \frac{\sin X}{X} \cdot \frac{\sin Y}{Y} (1+\cos\theta) \cdot \sin\phi$$

$$E_\theta = \frac{jk E_0 ab}{4\pi r} e^{-jkr} \left[\sin\phi \cdot \frac{\sin X}{X} \cdot \frac{\sin Y}{Y} (1+\cos\theta)\right] \rightarrow (26)$$

For $\theta = 0^\circ$

$$E_\theta = \frac{jkab E_0 \cdot e^{-jkr}}{2\pi r} \left[\sin\phi \cdot \frac{\sin X}{X} \cdot \frac{\sin Y}{Y}\right] \rightarrow (27)$$

similarly, substitute equs. (19) and (20) in (23)

$$E_\phi = -j\omega\left(A_y \cos\phi - \eta F_x \cos\theta \cos\phi\right)$$

$$E_\phi = -j\omega\left(-\mu H_0 G I \cos\phi - \eta\varepsilon E_0 G I \cos\theta\cos\phi\right)$$

$$= j\omega G I \cos\phi\left(\mu H_0 + \eta\varepsilon E_0 \cos\theta\right)$$

$$= j\omega G I \cos\phi\left(\mu \cdot \frac{E_0}{\eta} + \eta\varepsilon E_0 \cos\theta\right)$$

$$E_\phi = j\omega G I \eta\varepsilon E_0 (1+\cos\theta)\cos\phi$$

$$E_\phi = j\underbrace{\omega\eta\varepsilon}_{K} \cdot E_0 \cdot \frac{e^{-jkr}}{4\pi r} \cdot ab\left(\frac{\sin x}{x}\right)\left(\frac{\sin y}{y}\right)(1+\cos\theta)\cos\phi$$

$$E_\phi = jK E_0 \frac{ab\, e^{-jkr}}{4\pi r} \cos\phi\left(\frac{\sin x}{x}\right)\left(\frac{\sin y}{y}\right)(1+\cos\theta) \rightarrow (28)$$

For $\theta = 0°$,

$$E_\phi = jK E_0 \frac{ab\, e^{-jkr}}{2\pi r} \cdot \cos\phi\left(\frac{\sin x}{x}\right)\left(\frac{\sin y}{y}\right) \rightarrow (29)$$

$$H_\theta = -\frac{E_\phi}{\eta}$$

$$= -jK E_0 \frac{ab\, e^{-jkr}}{2\pi\eta r} \cdot \cos\phi\left(\frac{\sin x}{x}\right)\left(\frac{\sin y}{y}\right) \rightarrow (30)$$

$$H_\phi = \frac{E_\theta}{\eta}$$

$$= \frac{jKab E_0 \cdot e^{-jkr}}{2\pi\eta r} \cdot \sin\phi \cdot \left(\frac{\sin x}{x}\right) \cdot \left(\frac{\sin y}{y}\right) \rightarrow (31)$$

$$E_\theta = \frac{jabkE_0 e^{-jkr}}{2\pi r}\left[\sin\phi \cdot \frac{\sin\left(\frac{ka}{2}\sin\theta.\cos\phi\right)}{\left(\frac{ka}{2}\sin\theta\cos\phi\right)} \times \frac{\sin\left(\frac{kb}{2}\sin\theta\sin\phi\right)}{\left(\frac{kb}{2}\sin\theta\sin\phi\right)}\right] \rightarrow (32)$$

$$E_\phi = \frac{jabkE_0 e^{-jkr}}{2\pi r}\left[\cos\theta\cos\phi \cdot \frac{\sin\left(\frac{ka}{2}\sin\theta\cos\phi\right)}{\left(\frac{ka}{2}\sin\theta\cos\phi\right)} \times \frac{\sin\left(\frac{kb}{2}\sin\theta\sin\phi\right)}{\left(\frac{kb}{2}\sin\theta\sin\phi\right)}\right] \rightarrow (33)$$

The E-plane is on YZ plane $(\phi = \pi/2)$

H-plane is on XZ plane $(\phi = 0)$

$\therefore$ For E-plane : $(\phi = \pi/2)$:-

$$E_r = E_\phi = 0$$

$$E_\theta = \frac{jabkE_0 e^{-jkr}}{2\pi r}\left[\frac{\sin\left(\frac{kb}{2}\sin\theta\right)}{\left(\frac{kb}{2}\sin\theta\right)}\right] \rightarrow (34)$$

For H-plane $(\phi = 0)$:-

$$E_r = E_\theta = 0$$

$$E_\phi = \frac{jabkE_0 e^{-jkr}}{2\pi r}\left[\cos\theta \cdot \frac{\sin\left(\frac{ka}{2}\sin\theta\right)}{\left(\frac{ka}{2}\sin\theta\right)}\right] \rightarrow (35)$$

Beamwidths :-

Null Beamwidth :-

For E-plane, the maximum radiation is directed along the z-axis ($\theta = 0$)

$\therefore$ The nulls (zeros) occur when

$$\frac{kb}{2} \sin\theta \Big|_{\theta=\theta_n} = n\pi, \quad n = 1,2,3\ldots$$

or at an angles of

$$\theta_n = \sin^{-1}\left(\frac{2n\pi}{kb}\right)$$

$$\theta_n = \sin^{-1}\left(\frac{n\lambda}{b}\right) \text{ rad} \rightarrow (1) \quad n = 1,2,3\ldots$$

(or)

$$\theta_n = 57.3 \sin^{-1}\left(\frac{n\lambda}{b}\right) \text{ degrees} \rightarrow (2) \quad n = 1,2,3\ldots$$

$$K = \frac{2\pi}{\lambda}$$

$$\frac{2\pi n}{kb} = \frac{\cancel{2}n\cancel{\pi}}{\frac{\cancel{2\pi} b}{\lambda}}$$

$$\therefore \frac{2\pi n}{kb} = \frac{n\lambda}{b}$$

if $b \gg \lambda$, then

$$\theta_n \simeq \frac{n\lambda}{b} \text{ rad}$$

(or)

$$\theta_n \simeq 57.3\left(\frac{n\lambda}{b}\right) \text{ degrees. } \quad n = 1,2,3\ldots \rightarrow (3)$$

The total beamwidth Between nulls is given by

$$\boxed{\begin{aligned} \Theta_n = 2\theta_n &= 2\sin^{-1}\left(\frac{n\lambda}{b}\right) \text{ rad (or)} \rightarrow (4) \\ &114.6 \sin^{-1}\left(\frac{n\lambda}{b}\right) \text{ degrees}, \quad n = 1,2,3\ldots \end{aligned}}$$

if $b \gg n\lambda$ then

$$\theta_n \simeq \frac{2n\lambda}{b} \text{ rad} = 114.6\left(\frac{n\lambda}{b}\right) \text{ degrees}, \quad n=1,2,3\ldots \rightarrow (5)$$

First null Beamwidth (FNBW) is obtained by substituting $n=1$ in (5).

Half Power Beamwidth (HPBW):

The half power point occurs when

$$\frac{kb}{2}\sin\theta\Big|_{\theta=\theta_h} = 1.391 \rightarrow (6)$$

or at an angle of

$$\theta_h = \sin^{-1}\left(\frac{2.782}{kb}\right) = \sin^{-1}\left(\frac{0.443\lambda}{b}\right) \text{ rad} \rightarrow (7)$$

or $$\theta_h = 57.3\sin^{-1}\left(\frac{0.443\lambda}{b}\right) \text{ degree} \rightarrow (8)$$

if $b \gg 0.443\lambda$ then

$$\theta_h \simeq \frac{0.443\lambda}{b} \text{ rad} = 25.38\left(\frac{\lambda}{b}\right) \text{ degrees} \rightarrow (9)$$

$\therefore$ The total half power beamwidth (HPBW) is given by

$$\boxed{\begin{aligned}\Theta_h = 2\theta_h &= 2\sin^{-1}\left(\frac{0.443\lambda}{b}\right) \text{ rad (or)} \\ &114.6\sin^{-1}\left(\frac{0.443\lambda}{b}\right) \text{ degrees} \rightarrow (10)\end{aligned}}$$

if $b \gg 0.443\lambda$ then

$$\theta_h \simeq \frac{0.886\lambda}{b} \text{ rad (or) } 50.8\left(\frac{\lambda}{b}\right) \text{ degrees} \rightarrow (11)$$

First Side Lobe Beamwidth (FSLBW) :-

The maximum of the first side lobe occurs when

$$\left.\frac{kb}{2}\sin\theta\right|_{\theta=\theta_s} = 4.494 \rightarrow (12)$$

or at an angle of

$$\theta_s = \sin^{-1}\left(\frac{8.988}{kb}\right) = \sin^{-1}\left(\frac{1.43\lambda}{b}\right) \text{ rad (or)}$$

$$57.3 \cdot \sin^{-1}\left(\frac{1.43\lambda}{b}\right) \text{ degrees} \rightarrow (13)$$

If $b \gg 1.43\lambda$ then,

$$\theta_s \simeq 1.43\left(\frac{\lambda}{b}\right) \text{ rad} = 81.9\left(\frac{\lambda}{b}\right) \text{ degrees} \rightarrow (14)$$

The total beamwidth between first side lobes (FSLBW) is given by

$$\boxed{\begin{aligned}\Theta_s = 2\theta_s &= 2\sin^{-1}\left(\frac{1.43\lambda}{b}\right) \text{ rad (or)} \\ &\quad 114.6\sin^{-1}\left(\frac{1.43\lambda}{b}\right) \text{ degrees}\end{aligned}} \rightarrow (15)$$

If $b \gg 1.43\lambda$ then

$$\Theta_s = 2.86\left(\frac{\lambda}{b}\right) \text{ rad} = 163.8\left(\frac{\lambda}{b}\right) \text{ degrees} \rightarrow (16)$$

Side Lobe Level (FSLMM):-

The maximum of equ (16) at the first side lobe is given by

$$E_\theta = \frac{j\,abk\,E_0\,e^{-jkr}}{2\pi r}\left[\frac{\sin\left(\frac{kb}{2}\sin\theta\right)}{\left(\frac{kb}{2}\sin\theta\right)}\right] \rightarrow (16)$$

$$|E_\theta(\theta = \theta_s)| = \left|\frac{\sin(4.494)}{4.494}\right|$$

$$= 0.217 \text{ (or) } -13.26 \text{ dB} \rightarrow (17)$$

which is 13.26 dB down from the maximum of the main Lobe.

Directivity :-

The aperture is mounted on an infinite ground plane so, an alternate and simpler method is used to compute the radiated power.

The average power density is first formed by using the fields at the aperture and it is then integrated over the physical bounds of the opening.

$$H_a = -\hat{a}_x \frac{E_0}{\eta} \rightarrow (18)$$

$$P_{rad} = \oiint_S W_{rad}\cdot ds = \frac{|E_0|^2}{2\eta}\iint_{S_a} ds = \frac{ab\,|E_0|^2}{2\eta} \rightarrow (19)$$

The maximum radiation intensity (U_{max}) occurs toward $\theta = 0^\circ$ and it is equal to

(From equ 16)

$$U_{max} = \left(\frac{ab}{\lambda}\right)^2 \frac{|E_0|^2}{2\eta} \rightarrow (20)$$

∴ The maximum directivity.

$$D_0 = \frac{4\pi U_{max}}{P_{rad}} = \frac{4\pi}{\lambda^2} ab \rightarrow (21)$$

$$= \frac{4\pi}{\lambda^2} A_p \rightarrow (22)$$

$$= \frac{4\pi}{\lambda^2} . A_{em} \rightarrow (23)$$

Problem :-

1. A Rectangular aperture with a constant field distribution with $a = 3\lambda$ and $b = 2\lambda$ is mounted on an infinite ground plane. Compute

a) FNBW b) HPBW c) FSLBW d) FSLMM and

60° 25.6° 91.3° −13.26 dB

e) directivity in E-plane.

18.77 dB