Directivity of Broadside Array :-

The directivity in case of Broadside array is defined as,

$$D_{max} = \frac{\text{Maximum radiation Intensity}}{\text{Average radiation Intensity}}$$

$$= \frac{U_{max}}{U_0} \Rightarrow \frac{U_{max}}{U_{avg}} \rightarrow ①$$

$$U_0 = \frac{P_{rad}}{4\pi} = \frac{1}{4\pi} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} |E(\theta,\phi)|^2 \sin\theta \, d\theta \, d\phi \rightarrow ②$$

$$E(\theta,\phi) = \frac{1}{n}\left[\frac{\sin\frac{n\psi}{2}}{\sin\frac{\psi}{2}}\right] \rightarrow ③$$

Assuming 'd' is very small as compared to length of the array. We can approximate $\sin\frac{\psi}{2} \simeq \frac{\psi}{2}$

$$\therefore E(\theta,\phi) = \frac{1}{n}\left[\frac{\sin\frac{n\psi}{2}}{\frac{\psi}{2}}\right] \rightarrow ④$$

sub ④ in ②

$$U_0 = \frac{1}{4\pi} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \left[\frac{\sin\frac{n\psi}{2}}{\frac{\psi}{2}}\right]^2 \sin\theta \, d\theta \, d\phi$$

$$= \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^{\pi} \left[\frac{\sin n\psi/2}{n\psi/2} \right]^2 \sin\theta \, d\theta$$

$$U_0 = \frac{1}{\cancel{4\pi}_{\,2}} \times \cancel{2\pi} \cdot \int_0^{\pi} \left[\frac{\sin \frac{n}{2}\cdot \beta d \cos\theta}{\frac{n}{2} \beta d \cos\theta} \right]^2 \sin\theta \, d\theta \quad \rightarrow (5)$$

Let $z = \frac{n\beta d \cos\theta}{2}$

$$dz = -\frac{n\beta d}{2} \sin\theta \, d\theta$$

$$\sin\theta \, d\theta = \frac{dz}{-\frac{n\beta d}{2}}$$

when $\theta = \pi$, $z = -\frac{n\beta d}{2}$

$\theta = 0$, $z = \frac{n\beta d}{2}$

$$\therefore U_0 = \frac{1}{2} \int_0^{\pi} \left[\frac{\sin z}{z} \right]^2 \frac{dz}{-\frac{n\beta d}{2}}$$

$$U_0 = \frac{1}{2} \times \frac{2}{-n\beta d} \int_{\frac{n\beta d}{2}}^{-\frac{n\beta d}{2}} \left[\frac{\sin z}{z} \right]^2 dz \quad \rightarrow (6)$$

For large array, n is large hence $n\beta d$ is also very large.

$$U_0 = \frac{-1}{n\beta d}\int_{\infty}^{-\infty}\left[\frac{\sin z}{z}\right]^2 dz$$

$$= \frac{1}{n\beta d}\int_{-\infty}^{\infty}\left[\frac{\sin z}{z}\right]^2 dz \quad \rightarrow (7)$$

By integration formula, $\int_{-\infty}^{\infty}\left[\frac{\sin z}{z}\right]^2 dz = \pi$

$$\therefore U_0 = \frac{1}{n\beta d}\pi \Rightarrow \frac{\pi}{n\beta d} \quad \rightarrow (8)$$

$$D_{max} = \frac{U_{max}}{U_0} \quad \rightarrow (9)$$

$$U_{max} = 1 \text{ at } \theta = 90^\circ \quad \rightarrow (10)$$

substitute (8) and (10) in (9)

$$\therefore D_{max} = \frac{1}{\pi/n\beta d} \Rightarrow \frac{n\beta d}{\pi} \Rightarrow \frac{n \cdot \frac{2\cancel{\pi}}{\lambda} \cdot d}{\cancel{\pi}}$$

$$\Rightarrow \frac{2nd}{\lambda}$$

The total length of the array is given by

$$L = (n-1)d$$

$$\simeq nd$$

$$\boxed{\therefore D_{max} = 2\cdot\left(\frac{L}{\lambda}\right)} \quad \rightarrow (11)$$

Directivity of End fire array :-

The directivity of End fire array is given by,

$$D_{max} = \frac{U_{max}}{U_0} \rightarrow (1)$$

For Endfire array, $U_{max} = 1$ and

$$U_0 = \frac{\pi}{2n\beta d}$$

$$\therefore D_{max} = \frac{U_{max}}{U_0} = \frac{1}{\pi/2n\beta d} \Rightarrow \frac{2n\beta d}{\pi}$$

$$= \frac{2n \times \frac{2\cancel{\pi}}{\lambda} \times d}{\cancel{\pi}}$$

$$= \frac{4nd}{L} \rightarrow (2)$$